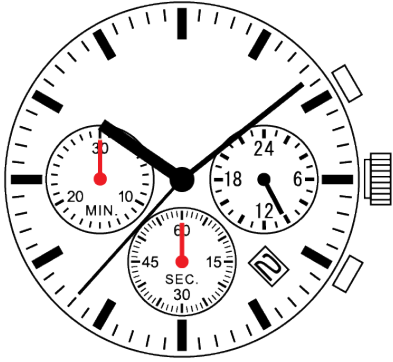


# Torsors and cocycles

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May 9, 2020

In my last post I mentioned that first Čech cohomology classes of a sheaf  $\mathcal{G}$  of (maybe non-abelian) groups admit a geometric interpretation in terms of  $\mathcal{G}$ -torsors. In this post I am going to introduce the notion of a  $\mathcal{G}$ -torsor over a topological space  $X$ , and show how the set of equivalence classes of  $\mathcal{G}$ -torsors on  $X$  can be identified with  $H^1(X, \mathcal{G})$ .



## The action groupoid

Before I talk about torsors, let me introduce a very simple concept, but that will be very useful in this and other posts. Take any left action  $G \times S \rightarrow S$  of a group  $G$  on a set  $S$ . To this action we can associate its *action groupoid*, which is the category  $[S, G]$  whose objects are precisely the elements of the set  $S$  and, for every  $x, y \in S$ , the set of morphisms from  $x$  to  $y$  is

$$\text{Mor}_{[S, G]}(x, y) = \{g \in G : g \cdot x = y\}.$$

This is clearly a groupoid since every morphism  $g$  is an isomorphism, with inverse given by  $g^{-1}$ . The *moduli set* of this category (that is, its set of isomorphism classes) is simply the set of orbits  $S/G$ .

In the last post we saw an example of a group action. Recall that for any open covering  $\mathfrak{U}$  of a topological space  $X$  and for any sheaf of groups  $\mathcal{G}$  over  $X$ , we had that an action of 0-cochains on 1-cochains by “conjugation”:

$$\begin{aligned} C^0(\mathfrak{U}, \mathcal{G}) \times C^1(\mathfrak{U}, \mathcal{G}) &\longrightarrow C^1(\mathfrak{U}, \mathcal{G}) \\ ((f_U)_{U \in \mathfrak{U}}, (g_{UV})_{U < V \in \mathfrak{U}}) &\longmapsto (f_U g_{UV} f_V^{-1})_{U < V \in \mathfrak{U}}. \end{aligned}$$

Moreover, we saw that this action respects cocycles. Now, as in the last post, we can use refinement maps to define the sets  $C^0(X, \mathcal{G})$ ,  $C^1(X, \mathcal{G})$  and  $Z^1(X, \mathcal{G})$  as limits by refinement. Suppose that we have some element of  $Z^1(X, \mathcal{G})$  represented by a pair  $(\mathfrak{U}, g)$ , with  $g \in Z^1(\mathfrak{U}, \mathcal{G})$  and some element of  $C^0(X, \mathcal{G})$  represented by another pair  $(\mathfrak{V}, f)$ . We can take a common refinement of both open covers by defining

$$\mathfrak{W} = \{U \cap V : U \in \mathfrak{U}, V \in \mathfrak{V}\}$$

(which is of course an open cover, since every point is in some  $U$  and in some  $V$ ) with refinement maps given by

$$\begin{aligned}\mathfrak{W} &\longrightarrow \mathfrak{U} \\ U \cap V &\longmapsto U\end{aligned}$$

(analogously for  $\mathfrak{W} \rightarrow \mathfrak{V}$ ). Thus, for the previously chosen elements we can take representatives  $g \in Z^1(\mathfrak{W}, \mathcal{G})$  and  $f \in C^0(\mathfrak{W}, \mathcal{G})$  and define

$$f \cdot g = (f_U g_{UV} f_V^{-1})_{U < V \in \mathfrak{W}}.$$

In conclusion, we have just defined an action of the group  $C^0(X, \mathcal{G})$  in the set  $Z^1(X, \mathcal{G})$  (of course, in the same way we can define an action on  $C^1$ , but we are particularly interested in this one). Moreover, the good properties of the direct limit guarantee that the set of orbits is precisely  $H^1(X, \mathcal{G})$ .

We can now consider the action groupoid  $[Z^1(X, \mathcal{G}), C^0(X, \mathcal{G})]$  associated to this action, whose moduli set is the Čech cohomology set  $H^1(X, \mathcal{G})$ . What we are going to do now is to give an interpretation of this action groupoid in terms of  $\mathcal{G}$ -torsors.

## Torsors

As above, let  $X$  be a topological space and  $\mathcal{G}$  a sheaf of groups over  $X$ .

**Definition 1.** A  $\mathcal{G}$ -torsor is a sheaf of sets  $\mathcal{F}$  on  $X$  endowed with an action  $\mathcal{G} \times \mathcal{F} \rightarrow \mathcal{F}$  such that

1. whenever  $\mathcal{F}(U)$  is nonempty, the action  $\mathcal{G}(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$  is free and transitive, and
2. for every  $x \in X$ , the stalk  $\mathcal{F}_x$  is nonempty.

A morphism of  $\mathcal{G}$ -torsors  $\mathcal{F} \rightarrow \mathcal{F}'$  is simply a morphism of sheaves compatible with the  $\mathcal{G}$ -actions (we say that it is  $\mathcal{G}$ -equivariant).

More precisely, given a morphism of  $\mathcal{G}$ -torsors  $\varphi : \mathcal{F} \rightarrow \mathcal{F}'$ , being  $\mathcal{G}$ -equivariant means that, if  $\mathcal{F}(U)$  is nonempty, for every  $p \in \mathcal{F}(U)$  we have

$$\varphi_U(g \cdot p) = g \cdot \varphi_U(p).$$

The best way to unravel this definition is by looking at **examples**. The simplest example of a  $\mathcal{G}$ -torsor is the *trivial  $\mathcal{G}$ -torsor*, which is  $\mathcal{F} = \mathcal{G}$  with the natural action given by the group operation. A key fact is now the following:

**Proposition 1.** *Let  $\mathcal{F}$  be a  $\mathcal{G}$ -torsor. If  $\mathcal{F}$  admits a global section, that is, if  $\mathcal{F}(X)$  is nonempty, then it is isomorphic to the trivial  $\mathcal{G}$ -torsor.*

*Proof.* Choose  $f \in \mathcal{F}(X)$ . Since  $f|_U \in \mathcal{F}(U)$  for every open subset  $U \subset X$ , the action  $\mathcal{G}(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$  is free and transitive. Therefore, every  $h_U \in \mathcal{F}(U)$  can be written in a unique way as  $h_U = g_U \cdot f|_U$ , for  $g_U \in \mathcal{G}(U)$ . Thus, the map

$$\begin{aligned}\mathcal{F}(U) &\longrightarrow \mathcal{G}(U) \\ h_U &\longmapsto g_U,\end{aligned}$$

which is clearly equivariant, defines a sheaf isomorphism. ■

Note now that since for every  $x \in X$ , we have that  $\mathcal{F}_x \neq \emptyset$ , there is an open cover  $\mathfrak{U}$  of  $X$  such that, for every  $U \in \mathfrak{U}$ , the set  $\mathcal{F}(U)$  is nonempty. Therefore, on every  $U \in \mathfrak{U}$ , the sheaf  $\mathcal{F}|_U$  is isomorphic to the trivial  $\mathcal{G}|_U$ -torsor. In conclusion, what property 2 in the definition of  $\mathcal{G}$ -torsor actually means is that every  $\mathcal{G}$ -torsor is, in some way, “locally trivial”. The open cover  $\mathfrak{U}$  is called a *trivializing cover*.

There are other examples of  $\mathcal{G}$ -torsors that the reader could be familiar with. To introduce these examples, first consider a topological group  $G$ . Associated to this group we can define two different sheaves. One is the *sheaf of  $G$ -valued functions*, which we denote simply by  $G$ , and is defined by

$$G(U) = \{\text{Continuous maps } U \rightarrow G\}.$$

The other one is the *sheaf of locally constant  $G$ -valued functions*, denoted by  $\underline{G}$ , and defined by

$$\underline{G}(U) = \{\text{Continuous maps } U \rightarrow G \text{ that are locally constant}\}.$$

Note that these two sheaves are essentially different, although they coincide if the group  $G$  is endowed with the discrete topology. Now,  $G$ -torsors are best known as *principal  $G$ -bundles* (or rather, as their sheaves of sections). On the other hand  $\underline{G}$ -torsors can be identified with  *$G$ -covering spaces*. I will say a lot about these two examples in future posts. If the reader is familiar with principal bundles, maybe it is useful for them to think of a  $\mathcal{G}$ -torsor as a generalization of a principal bundle in the sense that the structure group depends continuously on the base point.

Now, let us see how there is a groupoid naturally associated with  $\mathcal{G}$ -torsors:

**Proposition 2.** *Every morphism of  $\mathcal{G}$ -torsors is an isomorphism.*

*Proof.* Consider  $\varphi : \mathcal{F} \rightarrow \mathcal{F}'$  a morphism of  $\mathcal{G}$ -torsors. First, we will see that the map is injective. Suppose that there are  $p_1, p_2 \in \mathcal{F}(U)$  such that  $\varphi_U(p_1) = \varphi_U(p_2)$ . Since the action on  $\mathcal{F}(U)$  is transitive and free, there exists a unique  $g \in \mathcal{G}(U)$  such that  $p_1 = g \cdot p_2$  and, since  $\varphi$  is equivariant,  $\varphi_U(p_1) = g \cdot \varphi_U(p_2)$ . But the group  $\mathcal{G}(U)$  also acts freely and transitively on  $\mathcal{F}'(U)$ , so  $g = 1$  and  $p_1 = p_2$ . On the other hand, to see that it is surjective take any element  $p \in \mathcal{F}(U)$ . Since the action is transitive we can write any other element  $p' \in \mathcal{F}(U)$  as  $p' = g \cdot \varphi_U(p)$ , for some  $g \in \mathcal{G}(U)$ . Therefore, since  $\varphi$  is equivariant,  $p' = \varphi_U(g \cdot p)$ . ■

What we have just shown is that if we consider the category whose objects are  $\mathcal{G}$ -torsors and whose morphisms are morphisms of  $\mathcal{G}$ -torsors, this category is in fact a groupoid. The main purpose of this post is to show that this groupoid is equivalent to the action groupoid  $[Z^1(X, \mathcal{G}), C^0(X, \mathcal{G})]$ . In particular, this equivalence will yield a bijective correspondence between isomorphism classes of  $\mathcal{G}$ -torsors and cohomology classes in  $H^1(X, \mathcal{G})$ .

## Transition functions

The way to obtain a Čech cocycle from a  $\mathcal{G}$ -torsor is by considering *transition functions*. Consider a  $\mathcal{G}$ -torsor  $\mathcal{F}$  and a trivializing cover  $\mathfrak{U}$  of  $\mathcal{F}$ . Now, pick a section  $s_U \in \mathcal{F}(U)$  on each  $U \in \mathfrak{U}$  (I guess you need to use the Axiom of Choice here, but who cares –besides, we already used it to define cochains–). Now, for every two open sets  $U, V \in \mathfrak{U}$ , since the action of  $\mathcal{G}(U \cap V)$  on  $\mathcal{F}(U \cap V)$  is transitive, there must exist some cochain  $g = (g_{UV})_{U < V \in \mathfrak{U}} \in C^1(\mathfrak{U}, \mathcal{G})$  such that

$$s_U|_{U \cap V} = g_{UV} s_V|_{U \cap V}.$$

Moreover, this cochain is a cocycle since

$$g_{UV} g_{VW} s_W = g_{UV} s_V = s_U = g_{UW} s_W.$$

Thus, to any  $\mathcal{G}$  torsor  $\mathcal{F}$  we can associate a cocycle  $g \in Z^1(\mathfrak{U}, \mathcal{G})$  for some open cover  $\mathfrak{U}$  of  $X$ . This cocycle is called a *set of transition functions* of  $\mathcal{F}$ .

The choice of transition functions is not canonical, since it depends on the choice of the sections  $s_U$ . However, if we pick other sections  $s'_U \in \mathcal{F}(U)$  on each  $U \in \mathfrak{U}$ , since the action is transitive, we can write each  $s'_U$  as  $s'_U = f_U s_U$ , for some  $f_U \in \mathcal{G}(U)$ . Therefore, if we consider the cocycle  $g'$  defined by  $s'_U = g'_{UV} s'_V$ , we have

$$f_U s_U = s'_U = g'_{UV} s'_V = g'_{UV} f_V s_V,$$

so  $g'_{UV} = f_U g_{UV} f_V^{-1}$ . The same argument shows that if  $\varphi : \mathcal{F} \rightarrow \mathcal{F}'$  is a morphism of  $\mathcal{G}$ -torsors, and given a choice of the  $s_U$  and thus of the cocycle  $g$ , this cocycle and the cocycle  $g'$  determined by the  $\varphi_U(s_U)$  are related by a cochain  $f \in C^0(\mathfrak{U}, \mathcal{F})$  in the same way,  $g'_{UV} = f_U g_{UV} f_V^{-1}$ .

By choosing a trivializing cover for any  $\mathcal{G}$ -torsor and a set of transition functions, after taking the equivalence class in the direct limit we can define a morphism of groupoids by the following functor

$$\begin{aligned} \{\mathcal{G}\text{-torsors}\} &\longrightarrow [Z^1(X, \mathcal{G}), C^0(X, \mathcal{G})] \\ \mathcal{F} &\longmapsto \{\text{Transition functions of } \mathcal{F}\}, \end{aligned}$$

which maps any morphism of  $\mathcal{G}$ -torsors to the 0-cochain defined above.

**Proposition 3.** *This functor is an equivalence of categories. In particular, the set  $H^1(X, \mathcal{G})$  classifies isomorphism classes of  $\mathcal{G}$ -torsors.*

*Proof.* Clearly, the functor is fully faithful since the choice of open covering  $\mathfrak{U}$  and of  $f_U \in \mathcal{G}(U)$ , for  $U \in \mathfrak{U}$  determines  $\varphi$  as  $\varphi_U(s_U) = f_U s_U$ , for  $s_U \in \mathcal{F}(U)$ . Thus, it suffices to see that the functor is essentially surjective. This means that what we have to show is that given a cocycle in  $Z^1(X, \mathcal{G})$ , we can construct a  $\mathcal{G}$ -torsor whose transition functions are given by this cocycle. The way of doing this is a standard procedure which appears in a lot of places. The idea is to define the torsor locally as  $\mathcal{G}$  and then use the cocycle to “glue” the different patches. More precisely, we choose a representative  $(\mathfrak{U}, g)$ , with  $g \in Z^1(\mathfrak{U}, \mathcal{G})$ , of the chosen cocycle and define a presheaf

$$\mathcal{F}(U) = \coprod_{V \in \mathfrak{U}} \mathcal{G}(U \cap V) / \sim,$$

with the equivalence relation  $\sim$  given as follows. We say that two sections  $f \in \mathcal{G}(U \cap V)$  and  $f' \in \mathcal{G}(U \cap V')$ , with  $V \cap V' \neq \emptyset$ , are related if

$$f|_{U \cap V \cap V'} = g_{VV'} f'|_{U \cap V \cap V'}.$$

This presheaf verifies the sheaf condition by construction and it is a  $\mathcal{G}$ -torsor since on every  $U \in \mathfrak{U}$  it is the trivial  $\mathcal{G}$ -torsor. Again by construction, the cocycle  $g$  gives the transition functions of  $\mathcal{F}$ . ■

## A nice application

For the well known cases associated to  $G$  topological group, the above result is telling us that (isomorphism classes of) principal  $G$ -bundles are classified by the cohomology set  $H^1(X, G)$  and that  $G$ -covering spaces are classified by  $H^1(X, \underline{G})$ .

In a future post, I will explain how the correspondence between  $G$ -covering spaces and  $H^1(X, \underline{G})$  gives a nice and maybe “non-standard” approach at the basic results of Algebraic Topology. As for now, I am going to show how the correspondence between principal  $G$ -bundles and  $H^1(X, G)$  can be combined with the results of my last post to prove a nice fact of principal bundle theory.

What we are going to consider now is the problem of lifting the structure group to a group extension. In general, for any group  $G$  we say that another group  $\hat{G}$  is an *extension* of  $G$  if there is a surjective homomorphism  $\hat{G} \rightarrow G$ . More generally, if  $1 \rightarrow A \rightarrow \hat{G} \rightarrow G \rightarrow 1$  is a short exact sequence of groups, we say that  $\hat{G}$  is an *extension of  $G$  by  $A$* . Moreover, if the homomorphism  $A \rightarrow \hat{G}$  factors through the centre of  $\hat{G}$ , we say that the extension  $\hat{G}$  is a *central extension*. In particular, if  $\hat{G}$  is a central extension, the group  $A$  is abelian.

The lifting problem consists on, given a central extension  $1 \rightarrow A \rightarrow \hat{G} \rightarrow G \rightarrow 1$  and a principal  $G$ -bundle  $E$  over a topological space  $X$ , constructing a principal  $\hat{G}$ -bundle  $\hat{E}$  “lifting”  $E$ . In our terms, we can regard  $\hat{G} \rightarrow G$  as a morphism of sheaves, that induces a map  $H^1(X, \hat{G}) \rightarrow H^1(X, G)$ . What we want to know is when this map is surjective. Recall now from my last post that, since  $A$  is abelian,  $H^2(X, A)$  is defined and the short exact sequence  $1 \rightarrow A \rightarrow \hat{G} \rightarrow G \rightarrow 1$  induces in cohomology the exact sequence

$$H^1(X, \hat{G}) \rightarrow H^1(X, G) \rightarrow H^2(X, A).$$

Therefore, the map  $H^1(X, \hat{G}) \rightarrow H^1(X, G)$  is surjective if and only if  $H^2(X, A)$  is trivial.

**Example.** A nice example where this lifting problem is interesting is given by spin structures. Let  $X$  be an  $n$ -dimensional Riemannian manifold. Its tangent bundle  $TX$  is a vector bundle and, by considering its *frame bundle* we can regard it as a principal  $\mathrm{GL}(n, \mathbb{R})$ -bundle. Now, the Riemannian metric gives a reduction of the structure group to a principal  $\mathrm{SO}(n)$ -bundle. A *spin structure* on  $X$  is a lift of the structure group from this principal  $\mathrm{SO}(n)$ -bundle to the universal covering space  $\mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ . It is well known that  $\mathrm{SO}(n)$  is doubly connected. For example,  $\mathrm{SO}(3)$  is diffeomorphic to the real projective space  $\mathbb{R}P^3$  and  $\mathrm{Spin}(3) = \mathrm{SU}(2)$  is diffeomorphic to the 3-sphere  $\mathbb{S}^3$ . Therefore, the covering homomorphism  $\mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$  is in fact a central extension

$$1 \rightarrow \mathbb{Z}/(2) \rightarrow \mathrm{Spin}(n) \rightarrow \mathrm{SO}(n) \rightarrow 1.$$

We conclude from this that the obstruction for defining spin structures on  $X$  will be given by its Čech cohomology set  $H^2(X, \mathbb{Z}/(2))$ . If  $g \in H^1(X, \mathrm{SO}(n))$  denotes the cocycle associated to the tangent bundle, the element  $\delta(g) \in H^2(X, \mathbb{Z}/(2))$  is called the *second Stiefel-Whitney class* of  $X$ , denoted  $\omega_2(X)$ . We will be able to define a spin structure on  $X$  whenever this class vanishes,  $\omega_2(X) = 0$ .

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